Math 255A Lecture 19 Notes

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1 The Toeplitz Index Theorem

1.1 Hardy space

Let $H = \{u \in L^2((0, 2\pi)) : \hat{u}(n) = 0 \, \forall n < 0\} \subseteq L^2((0, 2\pi))$, where the Fourier coefficients are $\hat{u}(n) = (1/2\pi) \int_0^{2\pi} e(\theta) e^{-in\theta} \, d\theta$. If $u \in H$, then $u(\theta) \sim \sum_{n=0}^\infty \hat{u}(n) e^{in\theta}$ can be viewed as the boundary values of the holomorphic function $\sum_{n=0}^\infty \hat{u}(n) z^n$ with $|z| < 1$. The space H is called the **Hardy space**.

Let $\Pi : L^2((0, 2\pi)) \to H$ be the orthogonal projection sending $u \sim \sum_{n=0}^\infty \hat{u}(n) e^{in\theta} \mapsto \sum_{n=0}^\infty \hat{u}(n) e^{in\theta}$. Given $f \in L^\infty((0, 2\pi))$, associated to f is the **Toeplitz operator** $\operatorname{Top}(f) : H \to H$ sending $u \mapsto \Pi(fu)$. We have $\| \operatorname{Top}(f) \|_{\mathcal{L}(H,H)} \leq \|f\|_{L^\infty}$.

1.2 The Toeplitz index theorem

Theorem 1.1 (Toeplitz index theorem)**.** *Let f be continuous 2π-periodic, and assume that f has no zeros. Then* $\operatorname{Top}(f)$ *is Fredholm, and* $\operatorname{ind}(\operatorname{Top}(f)) = -$*winding number*$(f)$.

To define the winding number, write $f(\theta) = r(\theta) e^{i\varphi(\theta)}$ with $r > 0$ and $0 \leq \theta \leq 2\pi$. The winding number of f is $(\varphi(2\pi) - \varphi(0))/2\pi$. If $f \in C^1$, then the winding number of f is

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(\theta)}{f(\theta)} \, d\theta.$$

Proof. To establish the Fredholm property, we try to invert $\operatorname{Top}(f)$ modulo a compact error. Here is a claim: Let f, g be continuous 2π-periodic. Then $\operatorname{Top} \operatorname{Top}(g) = \operatorname{Top}(fg) +$ compact operator.

Write $\operatorname{Top}(f) = \Pi M_f$ and ΠM_g, where M_f, M_g are multiplication operators by f and g. Then $\operatorname{Top}(f) \operatorname{Top}(g) = \Pi M_f \Pi M_g = \Pi(\Pi M_f + [M_f \Pi]) M_g$, where $[M_f, \Pi] = M_f \Pi - \Pi M_f$ is the commutator. So we get

$$\Pi(\Pi M_f + [M_f \Pi]) M_g = \Pi M_f M_g + \Pi[M_f, \Pi] M_g = \operatorname{Top}(fg) + \Pi[M_f, \Pi] M_g.$$

It suffices to show that $[M_f, \Pi] : L^2 \to L^2$ is compact. We split into cases.

If $f(\theta) = e^{in\theta}$ with $n \in \mathbb{Z}$, $n \neq 0$, then if $n > 0$,

$$[M_f, \Pi]e^{ik\theta} = (M_f\Pi - \Pi M_f)e^{ik\theta} = \begin{cases} 0 & k \geq 0 \\ -\Pi(e^{i(k+n)\theta}) & k < 0. \end{cases}$$

Now observe that $-\Pi(e^{i(k+n)\theta}) = 0$ if $k < -n$, so the operator is of finite rank and is therefore compact. The computation is similar for $n < 0$.

If f is a trigonometric polynomial $f(\theta) = \sum_{-N}^{N} a_n e^{in\theta}$, then $[M_f, \Pi]$ is also of inite rank and is hence compact. If f is an arbitrary continuous, 2π-periodic function, let f_n be a sequence of trigonometric polynomials such that $f_n \to f$ uniformly. Then

$$\begin{aligned} \|[M_{f_n}, \Pi] - [M_f, \Pi]\| &= \|[M_{f_n} - M_f, \Pi]\| \\ &\leq \|M_{f_n - f}\Pi\| + \|\Pi M_{f_n - f}\| \\ &\leq 2\|f_n - f\| \to 0. \end{aligned}$$

Thus, $[M_f, \Pi]$ is compact.

So the claim holds. Now if $f \neq 0$, write $\operatorname{Top}(f)\operatorname{Top}(1/f) = I + \text{compact}$, and same for $\operatorname{Top}(1/f)\operatorname{Top}(f)$. So we get that $\operatorname{Top}(f)$ is Fredholm. Notice also that if f, g are continuous and nonvanishing, then $\operatorname{ind}(\operatorname{Top}(fg)) = \operatorname{ind}(\operatorname{Top}(f) + \operatorname{Top}(g) + \text{compact}) = \operatorname{ind}(\operatorname{Top}(f)) + \operatorname{ind}(\operatorname{Top}(g))$.

Now write $f(\theta) = r(\theta)e^{i\varphi(\theta)}$. Then we get $\operatorname{ind}(\operatorname{Top}(f)) = \operatorname{Top}(r) + \operatorname{Top}(e^{i\varphi})$. Take $t_t(\theta) = (1-t)r(\theta) = (1-t)r(\theta) + t > 0$ with $0 \leq t \leq 1$. To compute $\operatorname{ind}(\operatorname{Top}(e^{i\varphi}))$, write N for the winding number of f, and let $g_t(\theta) = e^{i(1-t)\varphi(\theta) + iNt\theta}$. Then g_t is periodic in θ and continuous in t. So $\operatorname{ind}(\operatorname{Top}(e^{i\varphi})) = \operatorname{ind}(\operatorname{Top}(e^{iN\theta})) = -N$. □